Jefferson County

Calculus Workshop

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Bruce MacMillan
University of Arizona Mathematics Department
and
Center for Recruitment and Retention of Math Teachers

bmacmillan@math.arizona.edu

I. Introduction to Limits

1. An important question: Given a function f, and a number a. As x gets closer and closer to a, but x does not equal a, does f(x) get closer and closer to some number L?

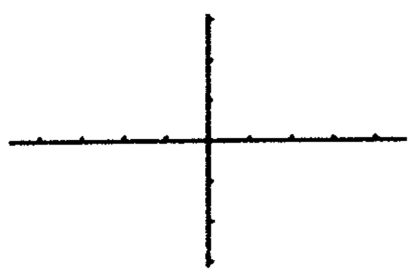
   If it does, then we say "the limit of f(x), as x approaches a, is equal to L", and we write $\lim_{x \to a} f(x) = L$.

   It is important to solve limit problems **numerically, graphically,** and **analytically**.

2. Example: Given $f(x) = \dfrac{x-1}{x^2+x-2}$. As $x \to 1$, does $f(x) \to$ _______?

On your calculator, set a "decimal window" (ZOOM, 4:ZDecimal) and let $Y_1 = \dfrac{x-1}{x^2+x-2}$.

a. Graph f and discuss the result.



b. Evaluate $Y_1(0.9)$, $Y_1(0.99)$, $Y_1(0.999)$, and $Y_1(1.1)$, $Y_1(1.01)$, $Y_1(1.001)$.

c. Graphically (and numerically) it appears that the $\lim_{x \to 1} \dfrac{x-1}{x^2+x-2} =$ _______.

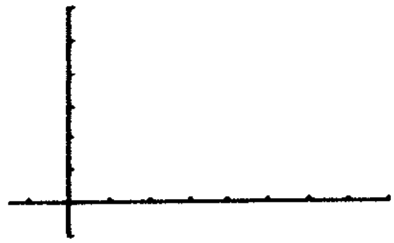
d. Evaluate $\lim_{x \to 1} \dfrac{x-1}{x^2+x-2}$ analytically.

$$\lim_{x \to 1} \frac{x-1}{x^2+x-2} =$$

e. What about $\lim_{x \to -2} \dfrac{x-1}{x^2+x-2}$? ____________________

3. Problem: Given $f(x)=(1+x)^{2/x}$

a. Sketch a graph of the function for $x > -1$. Show all asymptotes with dotted lines and other undefined values with an "open circle".



b. Estimate the $\lim_{x\to 0} f(x)$ by evaluating f for values close to 0. Approximate the limit to 4 decimal places.

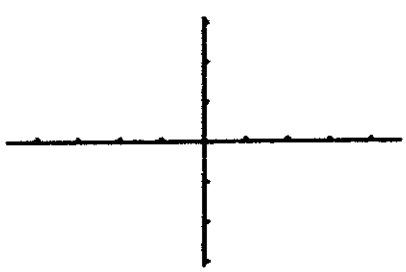
$$\lim_{x\to 0}(1+x)^{2/x} \approx \underline{\hspace{4cm}}$$

c. Do you know the **exact value** of $\lim_{x\to 0}(1+x)^{2/x}$? ________

II. Continuity

1. To develop the definition of continuity at a point, have students:

"Draw an example of a function that is not continuous (has a "break") at a number $x = c$. Draw as many different kinds of discontinuities at the number $x = c$ as you can."



Discuss what makes these functions discontinuous at the number $x = c$.

2. Worksheet - Limits and Continuity

3. My Favorite Function!

Consider the function $f(x) = x + [\![\cos(\pi x)]\!]$

Note: $f(x) = [\![x]\!]$ is called the "Greatest Integer Function" (or the "Floor Function") and can be found on most calculators as the **int(x)**.

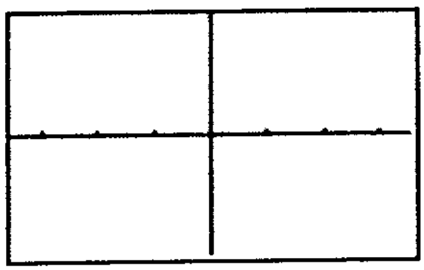
a. Sketch the graph of f using a Decimal Window, discuss the values of x that make f discontinuous, and explain, using the definition of continuous functions, why f is not continuous at these values.

III. The Derivative

1. Introduction to the Derivative - The Tangent Line Problem

Given $f(x) = x^2 - 3x$.

a. Sketch the graph of f. Verify with your calculator.



b. Find the <u>slope of the secant line</u> to $f(x) = x^2 - 3x$ passing through the points when $x = 1$ and $x = 3$.

c. Find the <u>slope of the tangent line</u> to $f(x) = x^2 - 3x$ passing through the point when $x = 1$.
(Note: We need to approximate the slope. How can we do that?)

On calculator, enter: $Y_1 = x^2 - 3x$

On Home Screen, enter $\dfrac{Y_1(1.01) - Y_1(1)}{1.01 - 1}$

Conjecture: As the "second point" gets closer to $x = 1$, the slope of the secant line approaches the slope of the tangent line. Therefore, we can write:

$$m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h} = f'(1)$$

d. Find the slope of the tangent line to $f(x) = x^2 - 3x$ at $x = 4$.

At $x = 4$, $m_{tan} =$ ________

e. Built into your calculator is a feature that will estimate the derivative of a function at a value. From the Home screen of your calculator, press the MATH key, and select 8:nDeriv(. (This stands for a "numerical derivative".)

The parameters for "nDeriv" are:

nDeriv(function , variable , value)

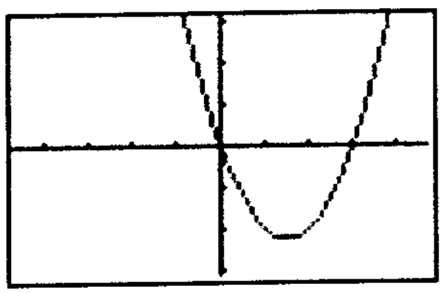
For our example, we should write: ________________________________

f. Use the “NDeriv” to find f’(a) for the values of a in the chart below if $f(x) = x^2 - 3x$

a	-1	0	1	2	3	4
f’(a)						

g. Do you see a pattern in the chart above? In other words, for any x, f’(x) = ______________

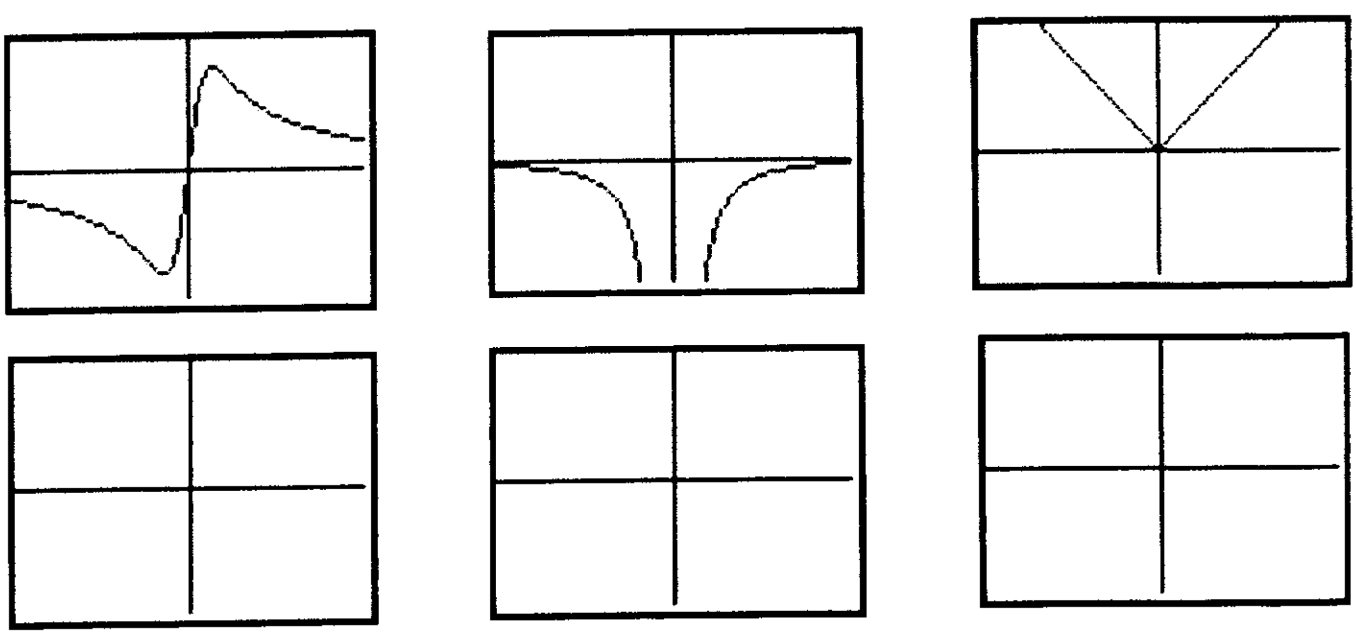
h. Sketch the graph of f’ with the graph of f. Does the graph of f give you any information about the graph of f’?



2. Graphs of f and f’.

Note: I like to introduce this concept <u>very early</u> in the development of the derivative concept. We are graphing derivative functions before we know any rules for finding derivatives analytically.

Given the graph of the function shown, sketch the graph of the derivative function f’ directly below it. Remember, **“the y value on the graph of f’ is the slope of the tangent line to the graph of f.”**



3. Worksheet - Graphs of f and f’

IV. Derivative Properties

1. Worksheet - Discovering Derivative Properties

2. Derivatives of Exponential Functions

Notes:

a. Traditionally, the $\frac{d}{dx}\left(e^x\right)$ is evaluated first, then generalized to the $\frac{d}{dx}\left(a^x\right)$.

I prefer to discover the $\frac{d}{dx}\left(a^x\right)$ first, then look at $\frac{d}{dx}\left(e^x\right)$ as a special case.

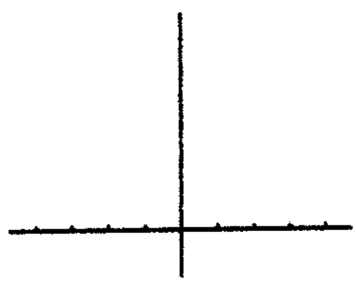
b. This is very difficult to prove analytically using the definition of the derivative. You get a limit expression such as:

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

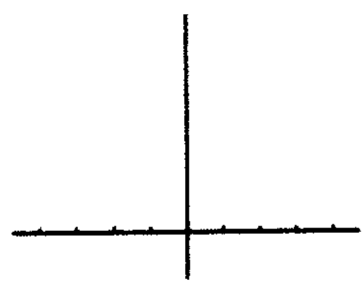
c. We should know that $\ln 1 = 0$, $\ln e = 1$, but need a calculator to evaluate ln2. $\ln 2 =$ ____________

d. If $f(x) = 2^x$, sketch the graphs of f(x) and f'(x).

f



f'



e. Let: $y_1 = 2^x$ .and $y_2 = \frac{d}{dx}\left(2^x\right)$. Graph the functions. (Note: Y2 = nDeriv(Y1 , X , X))

f. What is the relationship between y1 and y2? It looks like $y_2 = k \cdot y_1$, where $0 < k < 1$. To verify this, and to find k, let $y_3 = \frac{y_2}{y_1}$, and graph it.

k = ____________________

Therfore, $y_2 = \frac{d}{dx}\left(2^x\right) =$ ________________ = ________________

g. **Conclusions:**

1. $\frac{d}{dx}\left(2^x\right) =$ ____________   2. In general, $\frac{d}{dx}\left(a^x\right) =$ ______________

3. And, if $a = e$, then $\frac{d}{dx}\left(e^x\right) =$ ______________

3. Derivatives of Sine and Cosine functions (graphically).

4. Discovering the Chain Rule

Make a guess.

a. We know: If y=2sinx, then y'= ____________________

Guess: If y=2sin(5x+1), then y'= ____________________

b. We know: If $y = x^3$, then y'= ____________________

Guess: If $y = (3x^2 + 1)^3$, then y'= ____________________

c. We know: If $y = \sqrt{x}$, then y'= ____________________

Guess: If $y = \sqrt{\cos x}$, then y'= ____________________

Checking the guesses.

a. Check the guess for Part a above <u>graphically</u>.

Let: Y1= 2sin(5x+1)

Y2= ____________________ (The guess)

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

Was the guess correct? __________

The correct answer is: If y=2sin(5x+1), then y'= ____________________

b. Check the guess for Part b above algebraically.

If $y = (3x^2 + 1)^3$, expand the right side of the equation, then find the derivative.

c. If we see the pattern, correct the guess for Part c, and check the answer graphically.

If $y = \sqrt{\cos x}$, then y'=____________________________________

Let: $Y1 = \sqrt{\cos x}$

Y2= ____________________________ (New guess for y')

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

This property for finding derivatives of a composition of functions is called the **Chain Rule**.

It says:

If y = f(g(x)), then y'= ____

5. The Product Rule (Worksheet) and Quotient Rule

6. Worksheet - Discovering Derivatives of the Other Trigonometric Functions

7. Curve Sketching
It's always nice to have different and interesting functions to use for your "curve sketching" problems. Here are a few that I have found.

a. $f(x) = \sqrt[3]{x^2 - 2x}$ b. $f(x) = 2x^2e^x$

c. $f(x) = \sin^2 x$ d. $f(x) = \ln(\cos x)$ (Be careful of the domain!)

V. Applications of the Derivative

1. Calculus Application Problem - Heat It, Then Cool It
2. Calculus Application Problem - TICTOC
3. Calculus Application Problem - Keep On Folding
4. Calculus Application Problem - Around the Corner

VI. Implicit Differentiation

1. Introduce with the following problems:

a. $\frac{d}{dx}(3x + 4y) =$ ______________________________

b. $\frac{d}{dx}(3x^3 + 4y^2) =$ ______________________________

c. $\frac{d}{dx}(\cos x + \sin y) =$ ______________________________

2. A nice example to use when developing implicit differentiation.

Given $x^2 - 4y^2 = 16$

a. Do you know what this graph looks like? ____________________

b. Solve for y and graph the functions on your calculator.

c. Find y', and then find the slope of the tangent line at $\left(4\sqrt{2}, -2\right)$.

d. Use implicit differentiation to find y', and then find the slope of the tangent line at $\left(4\sqrt{2}, -2\right)$.

3. An example: Given the relation defined as $x^2 + xy + y^2 = 7$.

a. Use **implicit differentiation** to find the derivative y'.
(Hint: Don't forget to use the Product Rule on the "xy" term.)

b. Find the **slope of the tangent line** to the curve at the point (-2 , -1). ____________

c. Solve the original equation for y in terms of x and graph the function(s). Show your graph below.

(Hint: To solve for y, you need to use the quadratic formula.

Let a = _____ , b = _____ , c= _______)

d. Use your calculator (and your graph) to evaluate the derivative of the function that you graphed in part c above at the point (-2 , -1) to check your answer in part b.

e. Write the **equation of the tangent line** to $x^2 + xy + y^2 = 7$ at (-2 , -1). Graph the tangent line on your calculator and show the line on the graph above.

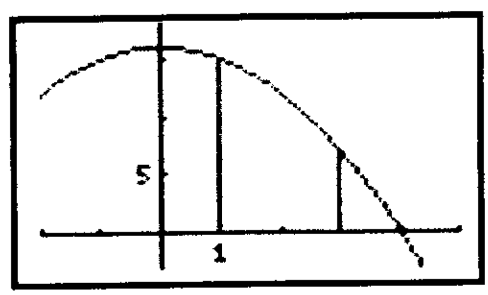
4. Other interesting implicitly defined relations.

a. $y^2(2-x)=x^3$ at (1 , -1)

b. $x^2y+xy^2=6$ at (1 , -3)

c. $2e^{xy}-x=0$ at (2 , 0)

VII. Introduction to the Definite Integral - The Area Problem



1. Example: Shown to the right is the region bounded by the function $f(x)=16-x^2$, the x-axis, and the lines $x = 1$ and $x = 3$.

Write and evaluate an expression we could use to approximate the area under $f(x)=16-x^2$ from $x = 1$ to $x = 3$ using:

a. 4 rectangles

b. 10 rectangles

c. 50 rectangles

d. n rectangles. (We can not evaluate this expression without a CAS, but it is important that we are able to write it.

e. How can we find the exact area of the region?

2. Another example: Given the region bounded by $f(x) = 2^x$, the x-axis, and the vertical lines $x = -2$ and $x = -4$.

a. Sketch the region described.

b. Write and evaluate (with your calculator) an expression that we could use to aproximate the area under $f(x) = 2^x$ from $x = -2$ to $x = 4$ using:

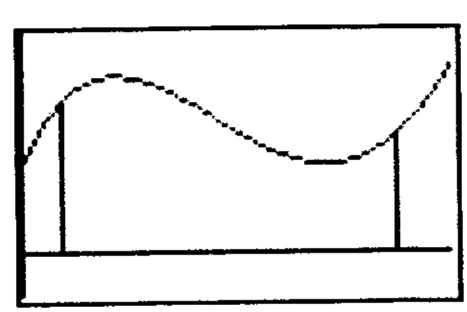
i. 6 rectangles, with their heights determined by the right endpoint of each subinterval (without using sigma notation). Show these 6 rectangles on your graph.

ii. 20 retangles with their heights determined by the right endpoint of each subinterval (using sigma notation).

iii. n rectangles (We can not evaluate this expression.)

3. Definition of the the Definite Integral - The Area Problem Generalized

Given a function f, such that $f(x) \geq 0$, and the lines $x = a$ and $x = b$. Write an expression in sigma notation that could be used to find the area of the region bounded by f, the x-axis, $x = a$ and $x = b$.



4. Evaluating Definite Integrals on the Calculator

This can be done one of two ways with our calculator.

a From the Home Screen, press the MATH key and select "9:fnInt". (This stands for a "function numerical integral".)

Then to evaluate $\int_{-2}^{4} 2^x\,dx$, enter:

fnInt(______________________________) = ______________

b. If the function has been graphed on your calculator with the interval included, from the graph select CALC, and 7: $\int f(x)dx$, and enter the limits of integration.

5 Program **DrawRec** on calculator

6. Worksheet - Evaluating Definite Integrals with Geometry

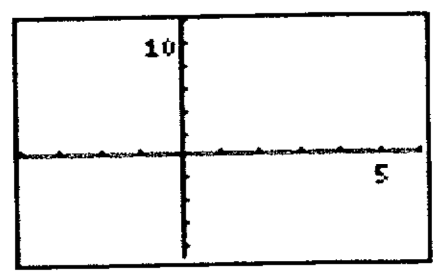
7. Worksheet - Properties of the Definite Integral

VIII. The Fundamental Theorem of Calculus

1. Introduction to the Fundamental Theorem of Calculus

Let $f(t) = -2t + 4$

Graph f on [-4 , 6].



Use geometry to evaluate the following integrals. Check your answer with your calculator.

a. $\int_{-2}^{-2} (-2t+4)dt =$ ______________

b. $\int_{-2}^{0} (-2t+4)dt =$ ______________

c. $\int_{-2}^{2} (-2t+4)dt =$ ______________

d. $\int_{-2}^{4} (-2t+4)dt =$ ______________

e. $\int_{-2}^{6} (-2t+4)dt =$ ______________

f. $\int_{-2}^{-4} (-2t+4)dt =$ ______________

Question: As the upper limit of integration in the definite integral changes, the value of the integral changes. Does this integral change values in some pattern that we can find?

Let $A(x)=\int_{-2}^{x}(-2t+4)dt$. Can we find a function rule for A(x)? We know that:

$A(-2)=0$, $A(0)=12$, $A(2)=16$, $A(4)=12$, $A(6)=0$, and $A(-4)=-20$

In general, given a function y=f(t). If $A(x)=\int_a^x f(t)dt$, can we find a function rule for A(x)?

2. Discovering the Fundamental Theorem of Calculus

a. Worksheet - Discovering the Fundamental Theorem of Calculus

b. Discussion: Where does the "constant" come from?

We know: $\int_{\frac{\pi}{2}}^{x}\cos t\,dt =$ ______________________________

Graph the integral $\int_{\frac{\pi}{2}}^{x}\cos t\,dt =$ using fnInt on your calculator.

Can you find where the constant C comes from? ______________________________

Therefore, $\int_{\frac{\pi}{2}}^{x}\cos t\,dt = =$ ______________________________

Conclusion: In general, let F be an antiderivative of f. If $A(x)=\int_a^x f(t)dt$, then

$A(x) =$ ______________________________

In the example at the very beginning of this discussion, we were trying to find a function A(x), such that $A(x)=\int_{-2}^{x}(-2t+4)dt$ and $A(-2)=0$, $A(0)=12$, $A(2)=16$, $A(4)=12$, and $A(6)=0$.

According to our conclusion, $A(x)=\int_{-2}^{x}(-2t+4)dt$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3.. The Fundamental Theorem of Calculus

If $g(x)=\int_{a}^{x}f(t)dt$, then $g'(x)=\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$

$=\frac{d}{dx}(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Part One of FTC: ________________________________

Also, if $\int_{a}^{x}f(t)dt=F(x)-F(a)$, where F is an antiderivative of f, and, if we substitute <u>b</u> for <u>x</u> we can write:

$\int_{a}^{b}f(t)dt = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

And if we substitute <u>x</u> for <u>t</u>, we can write:

$\int_{a}^{b}f(x)dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Part Two of FTC: $\int_{a}^{b}f(x)dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

4. Examples and Problems

Last class, we used the definition of the definite integral, and our calculator, to show that:

a. $\int_1^3 (16 - x^2)\,dx = 23.333\ldots$ Verify this.

b. $\int_0^\pi \sin x\,dx = 2$ Verify this.

Use the Fundamental Theorem of Calculus to evaluate the following definite integrals. Check your answers with your calculator.

c. $\int_1^9 \sqrt{x}\,dx$

d. $\int_0^1 (5e^x - 6x)\,dx$

IX. Activities/Problems with Area and Volume

1. Disk Method: $V = \pi \int_a^b (f(x))^2\,dx$

After deriving the "Disk Method" for finding the volume of solids of revolution, have your students prove familiar volume formulas.

Cylinder: $V = \pi r^2 h$ Cone: $V = \frac{1}{3}\pi r^2 h$ Sphere: $V = \frac{4}{3}\pi r^2$

TOP TEN STUDENT ERRORS
ON AP CALCULUS EXAMINATIONS

(Disclaimer: These are the opinion of Dan Kennedy and do not necessarily reflect the opinions of the College Board, the Educational Testing Service, the AP Calculus Test Development Committee, or even the results of sound statistical analysis. If you have taught AP Calculus for a while, however, you know in your heart that he is correct.)

1. $f''(x)=0 \Leftrightarrow (x, f(x))$ is a point of inflection.

2. $f(x)$ is a maximum (minimum) $\Leftrightarrow f'(x)=0$.

3. Average rate of change of f on $[a, b]$ is $\dfrac{f'(a)+f'(b)}{2}$.

4. Volume by washers is $\int_a^b \pi(R-r)^2\,dx$.

5. Separable differential equations can be solved without separating the variables.

6. Omitting the constant of integration, especially in initial value problems.

7. Graders will assume the correct antecedents for all pronouns used in justifications.

8. If the correct answer came from your calculator, the grader will assume your setup was correct.

9. Universal logarithmic antidifferentiation: $\int \frac{1}{f(x)}\,dx = \ln|f(x)| + C$.

10. $\frac{d}{dx} f(y) = f'(y)$ and other Chain Rule errors.

List of Problems used

1: 1969 MC AB17/BC17 ,1997 AB5/BC5 (d), 1995 BC5, 1998 MC AB 19
2: 2000 AB4 (d) and 1988 BC1 (c)
3: 1997 AB1 (b)
4: 1986 AB6/BC3 (b) and 1999 AB2/BC2 (b)&(c)
5: 1997 AB6/BC6 (a) and 2000 AB6 (a)
6: 2000 AB4 (a,c -- method 2) and 1989 BC1 (a)
7: 1996 AB1 (a,b,c)
8: 1998 AB5/BC5 (b,c,d)
9: 1993 MC AB22
10: 1992 AB4/BC1 (c) and 1993 MC AB24

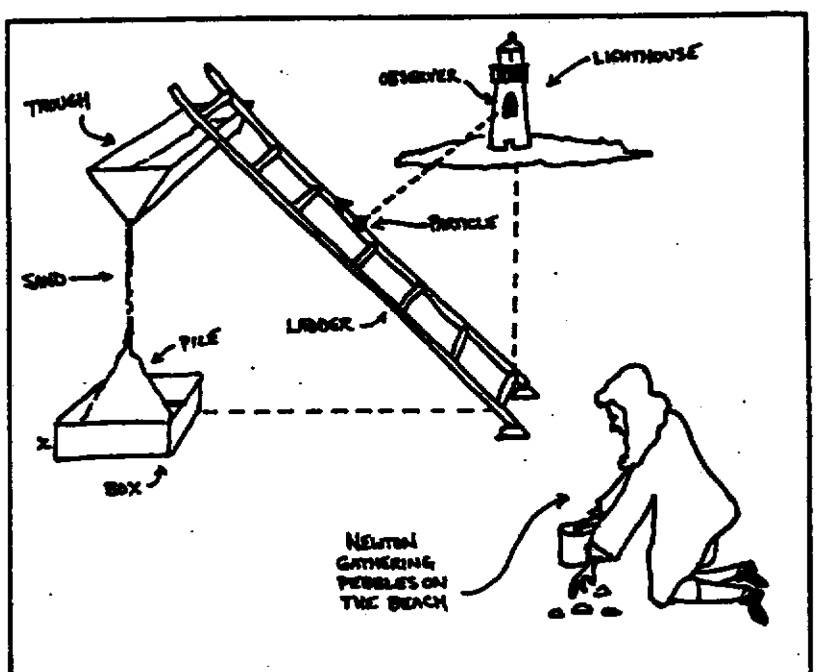
DAN KENNEDY

The All-Purpose Calculus Problem

Here's a calculus problem to end all calculus problems. (And you thought your professor assigned you hard ones!) See how many familiar themes you can find embedded in this problem.

A particle starts at rest and moves with velocity $v(t) = \int_1^t e^{-x^2}dx$ along a 10-foot ladder, which leans against a trough with a triangular cross-section two feet wide and one foot high. Sand is flowing out of the trough at a constant rate of two cubic feet per hour, forming a conical pile in the middle of a sandbox which has been formed by cutting a square of side x from each corner of an 8" by 15" piece of cardboard and folding up the sides. An observer watches the particle from a lighthouse one mile off shore, peering through a window shaped like a rectangle surmounted by a semicircle.

(a) How fast is the tip of the shadow moving?
(b) Find the volume of the solid generated when the trough is rotated about the y-axis.
(c) Justify your answer.
(d) Using the information found in parts (a), (b), and (c) sketch the curve on a pair of coordinate axes.



DAN KENNEDY is chair of the mathematics department at the Baylor School, Chattanooga, TN and is chair of the AP Calculus Committee.

Jefferson County

Calculus Workshop

Worksheets

Limits and Continuity

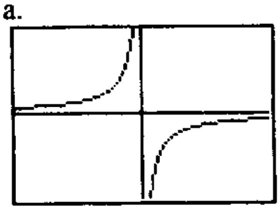
For the following problems, sketch a graph of a function that has the indicated features and write an equation for the function that has these features. The function may be a piecewise function.

1. The function is continuous at $x=3$, but has a cusp there.	2. The function has a limit as x approaches 3 but fails to be continuous there because $f(3)$ is undefined.
3. The function has a limit as x approaches -1, has a value for $f(-1)$, but still is not continuous there.	4. The function has no limit as x appoaches 0, but $f(0)=3$.
5. The function has a limit of 2 as x approaches 0 from the right, but has no limit as x approaches 0 from the left.	6. The function has a step (or jump) discontinuity at $x=1$, and $f(1) = 6$.
7. The function has a limit as x approaches 2 of 5, but $f(2) = 4$.	8. The function has a right-hand limit of -2 and a left-hand limit of 2 as x approaches -1.

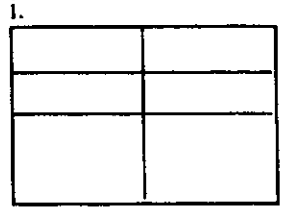
Graphs of f and f'

1. In the left column below are graphs of several functions. In the right-hand column - in a different order - are graphs of the associated derivative functions. Match each function with its derivative. (Note: The scales on the graphs are not all the same.)

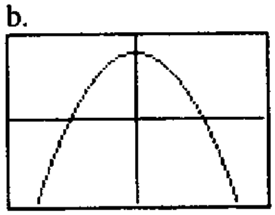
a.



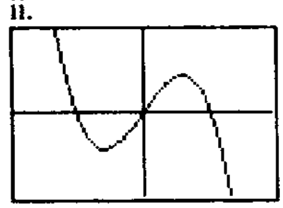
i.



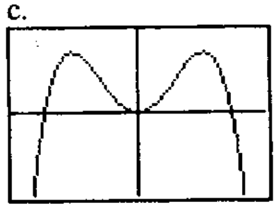
b.



ii.



c.



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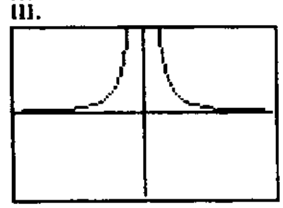

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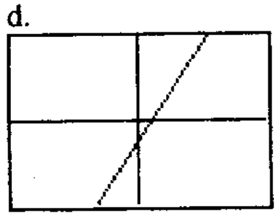

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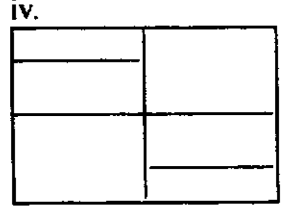

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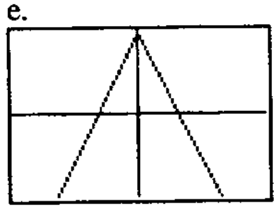

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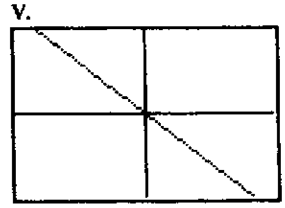


2.(a) Sketch a graph of the derivative of each function labeled (i) - (v) in the right column of the preceding problem.

(b) (Optional!) For each function labeled (a) - (e) in the left column of the preceding problem, sketch a graph of a function whose derivative is the function shown.

Discovering Derivative Properties

> For each of the following problems, a function **f(x)** is given. You are to try to discover the function **f'(x)** by finding a match for the calculator generated function **nDeriv(Y1,X,X)**.

1. Put the given function **f(x)** into **Y1** of your graphing calculator. (You may want to turn it off by deactivating it.)
2. Let **Y2 = nDeriv(Y1,X,X)**.
3. Guess the function that you see in **Y2** and check your guess by putting it into **Y3**.
4. If it matches, record your answer; if it doesn't, try again!
5. Don't forget, we are looking for patterns and generalizations that we can write as a property.

I. 1. $f(x) = x$ $f'(x) =$ ____________

2. $f(x) = x^2$ $f'(x) =$ ____________

3. $f(x) = x^3$ $f'(x) =$ ____________

4. $f(x) = x^4$ $f'(x) =$ ____________

> **Property (The Power Rule):**
>
> **If $f(x) = x^n$, then $f'(x) =$ ____________**

II. 5. $f(x) = 4x^2$ $f'(x) =$ ____________

6. $f(x) = -2x^3$ $f'(x) =$ ____________

7. $f(x) = 7x$ $f'(x) =$ ____________

> **Property:**
>
> **If k is a number and $f(x) = k\,g(x)$,**
>
> **then $f'(x) =$ ____________**

III. 8. $f(x) = 6$ $f'(x) =$ ____________

9. $f(x) = -3$ $f'(x) =$ ____________

> **Property:**
>
> **If k is a number and $f(x) = k$, then $f'(x) =$ ______**

IV. 10. $f(x) = -x^2 + 5x - 6$ $f'(x) =$ ____________

11. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6$,

$f'(x) =$ ________________

> **Property:**
>
> **If $f(x) = g(x) + k(x)$, then $f'(x) =$ ____________**

V. 12. $f(x) = \frac{1}{x} =$ ______ $f'(x) =$ ____________

13. $f(x) = \sqrt{x} =$ ______ $f'(x) =$ ____________

14. $f(x) = \sqrt[3]{x^2} =$ ______ $f'(x) =$ ____________

> **Question: Does the Power Rule hold for other numbers besides whole numbers?**
>
> **Negative whole numbers?** __________
>
> **Rational numbers?** __________

The Product Rule

Introduction: We already have a property that allows us to find the derivative of a **sum** (or difference) of two nctions.

Example:
If $\mathbf{f(x) = x^3 + x^5}$,
then $\mathbf{f'(x)}$ = ______ + ______

Property:
If $\mathbf{f(x) = g(x) + h(x)}$,
then $\mathbf{f'(x)}$ = ______ + ______

Objective: We want to discover a property that can be applied to find the **product** of two functions.

Let $\mathbf{g(x) = x^3}$ and $\mathbf{h(x) = x^5}$. Now, let $\mathbf{f(x) = g(x) \cdot h(x)}$ = __________ • __________ = __________.

Find: $\mathbf{f'(x)}$ = ______

$\mathbf{g'(x)}$ = __________

$\mathbf{h'(x)}$ = __________

$\mathbf{g'(x) \cdot h'(x)}$ = ______________

So, notice that although it is true that : **If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$,**
it is not true that: **If $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h'(x)$.**

It is possible to get the correct answer for $\mathbf{f'(x)}$ by a clever combination of the equations for $\mathbf{g(x)}$, $\mathbf{h(x)}$, $\mathbf{g'(x)}$, and $\mathbf{h'(x)}$.

Write the functions again.

(x) = ______, $\mathbf{h(x)}$ = ______, $\mathbf{f(x)}$ = ______, $\mathbf{g'(x)}$ = ______, $\mathbf{h'(x)}$ = ______, $\mathbf{f'(x)}$ = ______

(Remember, we are trying to get the function $\mathbf{f'(x)}$ from a combination of $\mathbf{g(x)}$, $\mathbf{h(x)}$, $\mathbf{g'(x)}$, and $\mathbf{h'(x)}$.)

See if you can figure out what this combination is (with the following hints)!

Notice that the **8** in $\mathbf{f'(x) = 8x^7}$ is the sum of the **3** and **5** in $\mathbf{g'(x) = 3x^2}$ and $\mathbf{h'(x) = 5x^4}$.

Fill in the following
$\mathbf{f'(x) = 8x^7 = 5x^7 + 3x^7}$ = __________ · $\mathbf{5x^4}$ + __________ · $\mathbf{3x^2}$.

Notice what the functions are that you put in the blanks!

Now try to complete the conjecture. This derivative property is called the **Product Rule.**

Product Rule: If $f(x) = g(x) \cdot h(x)$, then $f'(x)$ = ________ · ________ + ________ · ________

When you think you have it, try the following. If you don't have it, work at it!
Assume that your conjecture is true for the product of any two functions.

If $f(x) = f(x) = x^2 \cdot \ln x$, then $f'(x)$ = ________ · ________ + ________ · ________.

= ______________________ or ______________________

Check your result graphically by graphing f(x) in Y1, your derivative in Y2, and the nDeriv(Y1,X,X) in Y3.

Try one more.

If $f(x) = x^3e^{4x}$, then $f'(x) =$ ________ · ________ + ________ · ________.

= ____________________ or ____________________

Again, check your result graphically.

Examples of the Product Rule:

1. Let $f(x) = (3x^2 + 4)(2x^2 + 3)$

   a. Find $f'(x)$ by using the Product Rule.

   b. Find $f'(x)$ by expanding the terms first, then applying the Power Rule.

   c. Show that the two answers are equivalent.

2. Let $f(x) = x^2(x + 3)^2$

   a. Graph f on your calculator.

   b. From the graph of f, for what x-values does it appear that f has a horizontal tangent?

   x= ____________________

   c. Analytically, find all values of x where the graph of f has a horizontal tangent line.

Evaluating Definite Integrals with Geometry

Sketch the region whose area is given by the definite integral. Use geometry to evaluate the integral, then check your answer with your calculator. Note: Assume $\mathbf{k > 0}$ and $\mathbf{r > 0}$.

Examples:

1. $\int_{-1}^{3} 7\,dx$

2. $\int_{1}^{4} 2x\,dx$

3. $\int_{0}^{3} (-2x+6)\,dx$

4. $\int_{-3}^{3} (4-|x|)\,dx$

5. $\int_{0}^{3} \sqrt{9-x^2}\,dx$

A new geometric formula!

Archimedes discovered that the area of a **parabolic arch** can be found using the formula

$$\text{Area} = \frac{2}{3} \times \text{base} \times \text{height}$$

Find the area of the region bounded by $\mathbf{f(x) = 9 - x^2}$ and the x-axis.

Problems:

6. $\int_{-5}^{5} 4\,dx$

7. $\int_{a}^{b} k\,dx$

8. $\int_{2}^{10} \frac{1}{2}x\,dx$

9. $\int_{2}^{12} (8-\frac{1}{2}x)\,dx$

10. $\int_{-3}^{3} (|x|+2)\,dx$

11. $\int_{-3}^{3} |x+2|\,dx$

12. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

13. $\int_{-k}^{k} (k-|x|)\,dx$

14. $\int_{0}^{k} kx\,dx$

15. $\int_{0}^{r} \sqrt{r^2-x^2}\,dx$

16. $\int_{0}^{6} (6x-x^2)\,dx$

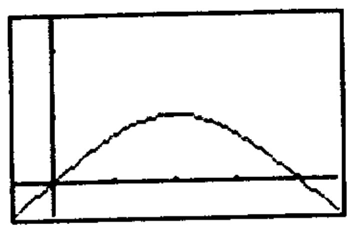
17. $\int_{-\sqrt{k}}^{\sqrt{k}} (k-x^2)\,dx$

Exploring Properties of Definite Integrals

Recall the definition of the definite Integral as a limit of a Riemann sum.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Let's begin with the definite integral: $\int_0^{\pi} \sin x dx$.



Guess, then evaluate $\int_0^{\pi} \sin x dx$ = ________

Use this result to guess, and then evaluate, the following definite integrals. Some of the results will lead to a property.

1. $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx =$ ________ Property: ________________

2. $\int_{\pi}^{0} \sin x dx =$ ________ Property: ________________

3. $\int_{\pi}^{2\pi} \sin x dx =$ ________

4. $\int_0^{2\pi} \sin x dx =$ ________ Property: ________________

5. $\int_0^{\pi} 2 \sin x dx =$ ________ Property: ________________

6. $\int_0^{\pi} 2 + \sin x dx$? ________ Property: ________________

Discovering the Fundamental Theorem of Calculus

Objective: Given a function $\mathbf{y = f(t)}$, find an "area function" $\mathbf{A(x)}$, defined to be $A(x)= \int_a^x f(t)dt$ which represents the area "under" **f** from **t=a** to **t=x**.

Instructions: Complete steps **a** through **d** below for the given functions and intervals.

a. Make a sketch of the function $\mathbf{y = f(t)}$ and shade the region on the interval given.

b. Use a <u>geometric formula</u> to find the area of the shaded region. Your answer should be a function of x and, in fact, be $A(x)= \int_a^x f(t)dt$. If necessary, simplify your answer by multiplying the terms.

c. Check your answer (function) by graphing. Enter into Y1, **Y1=fnInt(F(T) , T, a , x)** and into Y2, your answer for **A(x)**. If the two graphs are <u>not</u> the same curve, find your mistake and regraph.

d. Record your answer on this worksheet.

Examples: 1. $f(t) = 5$ on $[0, x]$. $A(x)= \int_0^x 5\,dt =$ ____________________

2. $f(t) = t$ on $[2, x]$. $A(x)= \int_2^x t\,dt =$ ____________________

Problems: 3. $f(t) = 3$ on $[-1, x.]$ $A(x)= \int_{-1}^x 3\,dt =$ ____________________

4. $f(t) = t + 3$ on $[0, x]$. $A(x)= \int_0^x (t+3)dt =$ ____________________

5. $f(t) = 4t$ on $[0, x]$. $A(x)= \int_0^x 4t\,dt =$ ____________________

6. $f(t) = 4t$ on $[1, x]$. $A(x)= \int_1^x 4t\,dt =$ ____________________

7. $f(t) = 2t + 3$ on $[0, x]$. $A(x)= \int_0^x (2t+3)dt =$ ____________________

8. $f(t) = 2t + 3$ on $[-1, x]$. $A(x)= \int_{-1}^x (2t+3)dt =$ ____________________

9. $f(t) = -3t + 7$ on $[-2, x]$ $A(x)= \int_{-2}^x (-3t+7)dt =$ ____________________

10. $f(t) = -3t + 7$ on $[1, x]$ $A(x)= \int_1^x (-3t+7)dt =$ ____________________

Conjecture: What is the relationship between the "area function" **A(x)** and the original function **f(t)**?

____

____

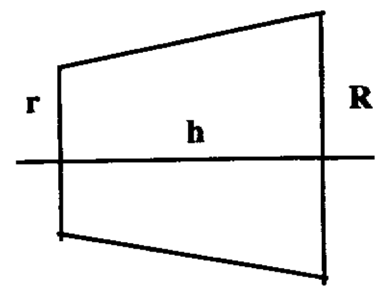
____

An Application of the Definite Integral - Cup Activity

In this activity you will use calculus techniques to solve a problem involving Volume and Surface Area. Your calculus will then be verified by using formulas from geometry.

I. Introduction:

A cup can be viewed as a "solid of revolution" by rotating a straight line segment about the x-axis as shown to the right. The only measurements we need to find are:



Radius of the top of the cup (R): ____________

Radius of the bottom of the cup (r): ____________

Height of the cup (h): ____________

In order to find both the volume and surface area of the cup we need to know the **equation of the line** formed by the **edge** of the cup. We can find it because we know two points on the line! The two points are:

(_______ , _______) and (_______ , _______).

Use these points and find the equation of the line. Show your work.

The equation of the line that represents the edge of the cup is: y = ____________________________

II. Volume:

1. Using calculus, find the **volume** of the cup (in cm^3). Write the integral that you are using and evaluate it using the **Fundamental Theorem of Calculus**. (You need to do a little algebra before finding the antiderivative.) Show all of your work.

Volume = ____________________ (You can check your answer using **fnInt** on your calculator.)

2. The volume of the cup can also be found using the geometry formula for the **volume of a cone** ($V = (1/3)\pi r^2 h$). To make a cone out of the figure above, extend the line segment of the edge of the cone so it crosses the x-axis. We need to know the **x-intercept**. Find it!

x-intercept: (________ , ________)

Now, the volume of the cup is the volume of the ***big cone*** minus the volume of the ***top cone***. Using the geometry formula above, write the expression for the volumes of the respective cones and evaluate it.

Volume = ____________________ - ____________________ = ____________________

Is this close to your calculus answer? How close? ____________________

III. Surface Area:

1. The **surface area** of a solid of revolution can be found by the complicated calculus formula:

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$ (where f(x) is the equation for the edge of the cup.)

Using calculus, find the surface area of the cup in cm^2. Show the integral that you are using and evaluate it **with your calculator**. (Not by hand!)

Surface Area = ____ = ____________________

2. The surface area of the cup can also be found using the geometry formula for the **surface area of a cone** (**$SA = \pi rL$** where **L** is the **slant height** of the cone). We need to know the **two** slant heights to use the surface area formula .
Find them! (Hint: The slant height is the **distance** between two points.)

Slant height of the ***big cone*** = ________________________________

Slant height of the ***top cone*** = ________________________________

Now, the surface area of the cup is the surface area of the ***big cone*** minus the surface area of the ***top cone***. Using the geometry formula above, write the two expression for the surface areas of the respective cones and evaluate it.

Surface Area = ____________________ - ____________________ = ____________________

Is this close to your calculus answer? How close? ________

IV. A Bonus Question:

If the "taper" of the sides of the cup remains the same, how **tall** (the height) should your cup be to be a "Big Gulp" and have a volume of 48 fluid ounces (which is approx 1420 cm^3)? Write an equation to solve this problem, and solve it graphically on your calculator!

Sketching "Area Functions"

Example :

Let $f(t) = 2t - 2$ on the interval $[-2, 4]$ and define $A(x) = \int_0^x (2t-2)dt$

a. Sketch a graph of **f** on the given interval.

b. Using your graph of **f**, sketch the graph of **A** on the same interval. (Note: The **y-value** of a point (x, y) on function **A** represents **area** in the graph of f from $t = 0$ to $t = x$.)

c. Verify your sketch in **part b** above by graphing $y = A(x)$ using **fnint** on your calculator.

d. Analytically, "guess" the rule for the function $y = A(x)$. Be sure to look at the important parts of the graph!

$A(x) =$ ______________________

e. Use the **Fundamental Theorem of Calculus** to derive the actual function for **A**.

f. How good was your guess?

Complete steps **a** through **f** above for the following functions on the given intervals. Be sure to answer all questions!

1. $f(t) = 3$ on $[-2, 3]$ and $A(x) = \int_0^x 3dt$

2. $f(t) = \frac{1}{2}t$ on $[-2, 3]$ and $A(x) = \int_1^x \frac{1}{2}t\,dt$

3. $f(t) = -t + 2$ on $[-1, 5]$ and $A(x) = \int_0^x (-t+2)dt$

4. $f(t) = t^2 - 2t$ on $[-2, 3]$ and $A(x) = \int_{-1}^x (t^2 - 2t)dt$

5. $f(t) = 4\sin 2t$ on $[-\pi, 2\pi]$ and $A(x) = \int_0^x 4\sin 2t\,dt$

Note: The amplitude of f is ________ and the period of f is __________

Answer the following questions about the relationship between the function **f(t)** and $A(x) = \int_a^x f(t)dt$.

1. What is the value of **A(a)**? __________

2. A point on **A** where a **maximum** or **minimum** occurs is a ________________ of **f**.

3. An **inflection point** on **A** occurs at a ________________ of **f**.

The FTC (One More Time)

Let a continuous and differentiable function $y = f(t)$ contain the following ordered pairs:

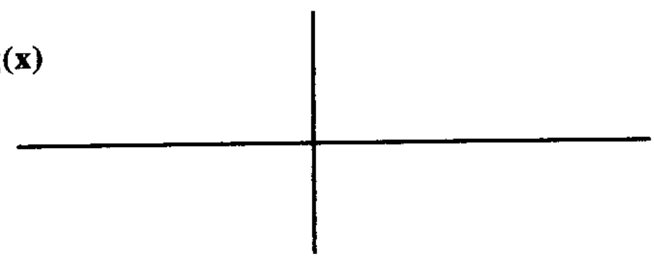
t	-4	-3	-2	-1	0	1	2	3	4
f(t)	0.06	0.1	0.2	0.5	1	0.5	0.2	0.1	0.06

Make a **scatterplot** of the data, with t values in L1 and f(t) values in L2. Also sketch the points and connect them to show $y = f(t)$ below. (Remember f(t) is differentiable.)

f(t)

If $g(x) = \int_0^x f(t)\,dt$, without your calculator sketch **g(x)** below.

g(x)



Where is **g(x)** concave up? ______________ What is true about **f(t)** on this interval? ______________

Where is **g(x)** concave down? ______________ What is true about **f(t)** on this interval? ______________

There are no maximum or minimum values on the graph of **g(x)**. Using the graph of **y = f(t)**, explain why this is true. ______________________________

Does it appear **g(x)** has horizontal asymptotes? __________

g(x) is a function that we know! Can you find it? Make a guess of g(x). **g(x) =** ______________

It is hard to tell if your guess for **g(x)** is correct, but, if it is, what would **g'(x)** be? **g'(x) =** ______________

If **g(x)** was correct, the graph of **g'(x)** should go through the points you originally graphed in your scatterplot. Try it.

This means that $\mathbf{f(t)}$ = ______________________________.

Recall that one part of the Fundamental Theorem of Calculus says that: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Complete **the Fundamental Theorem of Calculus** using the appropriate functions in this problem.

$$\frac{d}{dx}\int_a^x \underline{\hspace{4cm}}\,dt = \underline{\hspace{4cm}}$$

What is $\mathbf{g'(-3)}$? _______________ $\mathbf{g'(2)}$? _____________ These values can be found two different ways.

Explain.

1. ____

2. ____

Introduction to Slope Fields

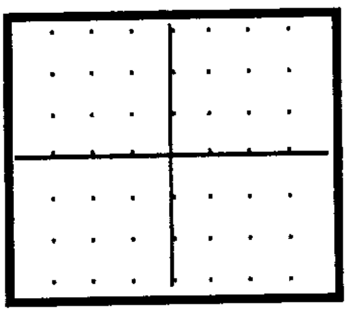
For each differential equation given below:

1. Plot the **slope field** on the grid provided.

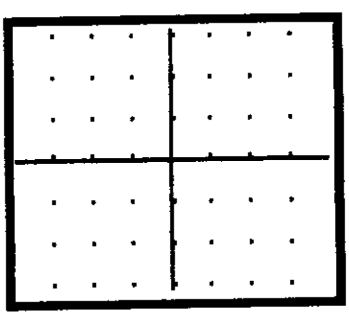
2. Suppose that you know that the point given is on a particular solution of the differential equation. By following the slopes, draw on your slope field what you think the **particular solution** looks like. (Note: The graph should follow the pattern of the slope field but may go between the points rather than through them.)

3. Solve the differential equation **algebraically**. Find the particular solution that contains the point given. Does your solution make sense to your graph of the slope field?

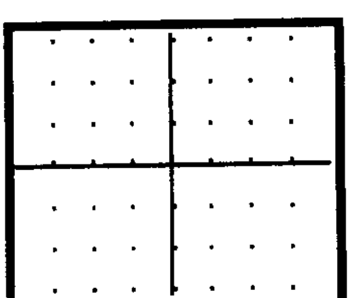
a. $\frac{dy}{dx} = -x$, $(-2, -1)$



b. $\frac{dy}{dx} = \frac{1}{x}$, $(1, 1)$



c. $\frac{dy}{dx} = -\frac{x}{y}$, $(2, -2)$



Jefferson County

Calculus Workshop

Calculus Application Problems

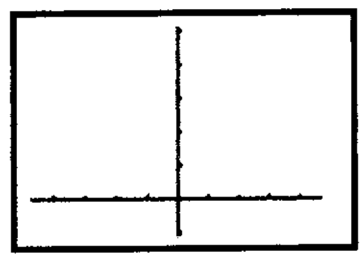
Calculus Application Problem - Heat It, Then Cool It!

Introduction: A temperature probe is placed in a cup of hot water. It remains there for approximately 35 seconds, then it is removed from this cup and placed in a cup of cold water for another 35 seconds. The objective of this activity is to find an algebraic function that models the temperature recorded by the probe over the entire 70 seconds, then apply some calculus concepts to the function.

An Algebra Review of Piecewise Functions:

A piecewise function is a function that is defined differently over different values of the domain. Two examples are presented. In the first, a piecewise function is given, and you are to graph it. In the second, the graph of a piecewise function is shown, and you are to write its rule.

1. Given the function $f(x)=\begin{cases} -x+2 & \text{if } x\le 1 \\ \sqrt{x} & \text{if } x>1 \end{cases}$

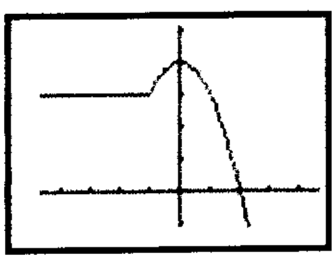


   a. Evaluate: $f(0) =$ ______

   $f(4) =$ ______

   b. Sketch the graph of f.

2. Write a rule for the piecewise function shown below.



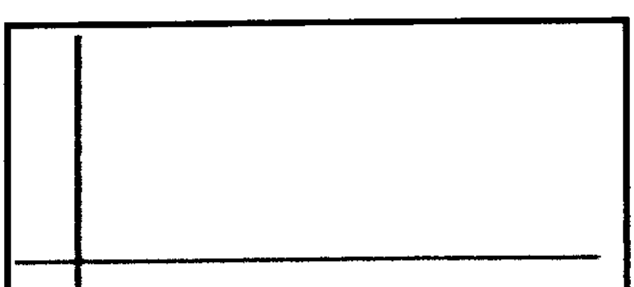
$f(x)=\begin{cases} \\ \end{cases}$

---

1. Complete the chart below from the data that was collected from the experiment. The data appears to have **horizontal asymptotes** at y = __________ and y = __________

Time	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70
Temp															

2. Create a scatterplot of the data on your calculator (**Time** in **L1** and **Temperature** in **L2**). Sketch the scatterplot below.



3. The General Solution (Newton's Law of Cooling/Heating)

We want to find a function **F(t)** that models the Fahrenheit temperature **F** of the probe at any time **t**, measured in seconds. Using a property of physics, called **Newton's Law of Cooling/Heating**, the temperature in an activity such as this can be modeled by an exponential function in the form:

$$\mathbf{F(t) = a \cdot b^{t} + c}.$$

4. The Specific Solution

First, to find an algebraic function that models our temperature vs. time data, it is clear that we need to write a piecewise function, one rule for the first (approximately) 35 seconds, and another for the last 35 (approximately) seconds. To find both of these rules, we will use the property of Newton's Law of Cooling.

In order to find a model of the form $\mathbf{F(t) = a \cdot b^{t} + c}$ for the first part of our data, we need to find the constants **a**, **b**, and **c**. We can find the constants **a** and **c** from the chart above (or by "tracing" on our scatterplot).

First we will find the constant **c**. According to Newton's Law of Cooling/Heating, and the data collected, the value of **c** would be approximately ____________.

To find **a**, record the temperature when $t = 0$. (0 , ________) Substitute this ordered pair, with the value of **c**, into our model $\mathbf{F(t) = a \cdot b^{t} + c}$, and solve for **a**. Show your work below.

a= ____________

To find **b**, the last constant in the model, we need another ordered pair. Let's use the ordered pair from the chart when $t = 10$ seconds.

Record this ordered pair. (10 , ________)

Substitute these values into the equation (with the values of **a** and **c**), and solve for the last unknown constant **b**. Show your work below. Round **b** to three decimal places. Then write the function **F(t)**.

b= ____________

F(t)= __________________

To check your work, graph your equation with your scatterplot to see how it fits the first part of the data. If it doesn't fit well, find your mistake!

The equation that fits the second part of the data is similar to the first equation. However, since we are beginning with a time other than $t = 0$, we need to apply a "horizontal shift" to our function. Therefore, the resulting form of this function is $\mathbf{F(t) = a \cdot b^{t-h} + c}$, where $\mathbf{h = 35}$. Use this form, and the hints given for the first part of the function, and find a rule that fits this part of the data in the scatterplot. Show all of your work below. Again, graph this equation to check it.

a= ____________

b= ____________ (3 decimals)

c= ____________

F(t)= __________________

F'(t) = $\left\{ \begin{array}{l} \\ \\ \end{array} \right.$

Verify your answer by graphing it on your calculator and comparing it to the graph drawn in part b above.

d. Using the function **F'(t)** above, how fast is the temperature changing when **t = 0, t = 10, and t = 45**? State the units with your answer.

F'(0) = ________________ **F'(10)** = ____________________ **F'(45)** = ____________________

e. An important calculus property, called the **Mean Value Theorem** (used in proving many calculus properties), says that on the domain of a function where the graph is "smooth"; i.e. no breaks or cusps, there must exist "at least one t-value where the instantaneous rate of change is equal to the average rate of change".

i. Consider the the time interval [0 , 15]. What is the <u>average rate of change</u> of **F(t)** on this interval? Show the expression below that you are using in evaluating this value.

ii. Since the derivative of a function represents the <u>instantaneous rate of change</u> of the function at any t-value, set the derivative **F'(t)** equal to the value found in part i above, and solve for **t**. Show your work!

iii. Now consider the the time interval [40 , 50]. What is the <u>average rate of change</u> of **F(t)** on this interval? Show the expression below that you are using in evaluating this value.

iv. Find the t-value where the <u>instantaneous rate of change</u> is equal to the <u>average rate of change</u> on [40 , 50]. Show your work!

Data Collection Activity - TICTOC

Objective: In this activity we are going to represent the motion of a pendulum with a sinusoidal function. We will then apply some calculus concepts to this function and its graph to further understand topics such as velocity, acceleration, etc.

Building the Function

The data we have collected and graphed is a distance (in L2) vs. time (in L1) scatterplot of the racquetball as it swings back and forth. It appears to produce a sinusoidal pattern, so we will attempt to fit our data with the sinusoidal function **d(t)** in the form:

$$d(t) = A\cos(B(t - C)) + D$$

We need to find values for **A**, **B**, **C**, and **D**. (If necessary, round all values to 2 decimal places.)

1. Find the value of **C** first. Since we are using a cosine function, trace to the point on the scatterplot where the value of **C** can be found and record it.

C= ________________

2. Now find the value of **D**. This should be the **average** of the "highest value" on the scatterplot and the "lowest value". By tracing, find these values, and show the arithmetic used to find **D**.

D= ________________

3. Now find the value of **A**. You already have all of the information to find **A**.

A= ________________

4. Finally, find the value of **B**. How many periods of the cosine function are on your scatterplot? _____
To find an accurate value for **B**, trace to the initial point of one period of the cosine function (which, actually, is the value of **C**) and then to the final point of the last period of the function. You should be able to find one period of the function if you subtract these two values and divide by the number of periods.

Length of one period: __________

Now you can find **B**. (Don't forget you need to use the number 2π to find **B**.) Record the value of **B**.

B= ____________

5. So, record the final **cosine function d(t)** that fits your data.

d(t) = ______________________________________

Enter this function into **Y1** of your calculator and graph it. If it doesn't fit the data, find your mistake!

Velocity and Acceleration Functions

6. We also have the "velocity" data in L3 of our calculator. Set up and graph a scatterplot of the velocity vs. time data. This scatterplot shows the velocity of the racquetball at a specific time as it swings back and forth.

7. We could determine this function using the same procedure as we did for the distance vs. tme data above, but we know calculus!

8. Determine the velocity function **v(t)** with calculus.

v(t) =

9. We also have the "acceleration" data in L4 of our calculator, but, from experience, we know that this might not be very accurate. Set up and graph a scatterplot of the acceleration vs. time data.

10. Again, we can find a function that fits this data using calculus. Determine the acceleration function **a(t)** with calculus.

a(t) = ________________________________

Check your answer by graphing it with the scatterplot. If it doesn't fit the data, find your mistake!

Questions

11. Turn off your scatterplots. Set the window of your calculator so you are seeing **exactly two periods** of the velocity and acceleration functions. Sketch the two periods of **v(t)** and **a(t)** below. Label the graphs and label the important points.

Let's use the graphs to answer some questions about the pendulum.

12. Over what time intervals is the pendulum moving **toward** the CBR? ____________________

13. When is the pendulum moving the fastest? **t** = ________________

14. What is occurring on the acceleration function at these points where the pendulum is moving the fastest?

____

The Speed Function

15. The **speed** of the pendulum can be represented by the **absolute value of the velocity function.** Enter a function for | (**v(t)** | and graph it. (Note: You may want to temporarily deactivate v(t) and a(t)).

16. What an interesting graph! The pendulum is speeding up when the "speed function" is increasing. Over what time intervals is the pendulum speeding up?

____

17. Now, let's see what the velocity function and the acceleration functions are doing on these time intervals. Deactivate the speed function and activate the velocity and acceleration functions. Rewrite the intervals above and tell whether the velocity and acceleration functions are positive (+) or negative (-) at these times.

Interval: __________ Velocity: __________ Acceleration: __________

Interval: __________ Velocity: __________ Acceleration: __________

Interval: __________ Velocity: __________ Acceleration: __________

18. The pendulum is slowing down when the "speed function" is decreasing. Activate the "speed function" again. Over what time intervals is the pendulum slowing down?

____

19. Deactivate the speed function and activate the velocity and acceleration functions one more time. Rewrite the intervals above and tell whether the velocity and acceleration functions are positive or negative at these times.

Interval: ____________ Velocity: __________ Acceleration: ____________

Interval: ____________ Velocity: __________ Acceleration: ____________

Interval: ____________ Velocity: __________ Acceleration: ____________

Interval: ____________ Velocity: __________ Acceleration: ____________

What are the patterns that you see? ______________________________________

____

____

A Bonus Problem!

20. Finally, let's look at one more relationship. Set up and graph a scatterplot between the "distance" data in L2, and the "velocity" data in L3. This result will be a surprise!

21. This type of relationship was discussed in class. By tracing (and using a little arithmetic), find the model that fits this data. To see if it is correct, solve for **y** in terms of **x** and graph it with the scatterplot. Show all of your work below. If it doesn't fit the data, find your mistake!

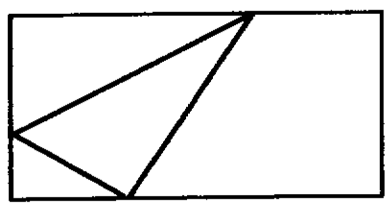
Calculus Application Problem - Keep on Folding!

The problem we are going to solve is:

> **Given a sheet of paper 8.5" by 11", fold the top left corner down to a point on the bottom edge. Where should you place this corner to maximize the area of the triangle formed in the bottom left corner?**

Remember, the <u>area of a triangle</u> can be found by the geometry formula: **Area =** ______________________.

This formula tells us that the area is a function of the __________ and the ___________ of the triangle. Let's do some measuring (and calculating!) and determine the area for some triangles. Take your 8.5" by 11" piece of paper and (holding it sideways) mark the bottom edge in 1" units and the left side of the paper in 0.5" units. (You only need to mark the bottom edge up to 8".)



Fold the top left corner down to the point on the bottom edge that measures 1". Estimate the height of the triangle to the nearest 0.1". ________ Now calculate the area of the triangle. __________. We are now going to let the base of the triangle grow longer.

What will happen to the height of the triangle? ____

What do you think will happen to the area of the triangle? ____

Let's see if the above conjectures are correct. Fold the top left corner down to the indicated point on the bottom edge of the piece of paper. Estimate the height of each triangle (to the nearest 0.1 in). Then calculate the area of the triangle. Use the chart below to show your results.

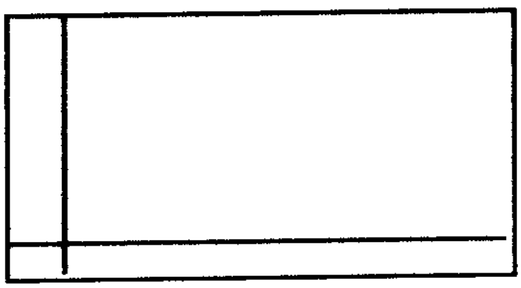
BASE	HEIGHT	AREA
1		
2		
3		
4		
5		
6		
7		
8		

From your chart where does it appear that the maximum value of the area of the triangle occurs? The maximum area of the triangle is ___________. It occurs when the base is __________ and the height is _____________.

Solutions to the Problem:

We want to determine the "exact value" of the base that will <u>maximize the area of the triangle</u>. We will do this two ways. First, a <u>numerical/graphical solution</u> (which does not require calculus), and then an <u>analytical solution</u> (which will require our calculus).

1. The Numerical/Graphical Solution
Let's look at the BASE, HEIGHT, and AREA data in graphical form. Put the values of the BASE in L1, the values of the HEIGHT in L2, and the values of the AREA in L3. We will first find a relationship between the BASE and the HEIGHT. Create a scatterplot of the BASE vs. HEIGHT (or L1 vs. L2). Set up your window to get a good view of your data points and plot the data. Draw a diagram of the data points below and indicate the window you used on your calculator.



[Xmin,Xmax]: ___________

[Ymin,Ymax]: ___________

What type of function do you think your data matches the closest? (It's not linear!) ____________________.
Using the regression option on your calculator, see if your guess was correct. Graph your regression equation.
How does it fit?__________________ Write the results of the regression equation below. (Round values to 2 decimal places.) Do not write the equation in terms of x and y, but in terms of **B** (BASE) and **H** (HEIGHT).

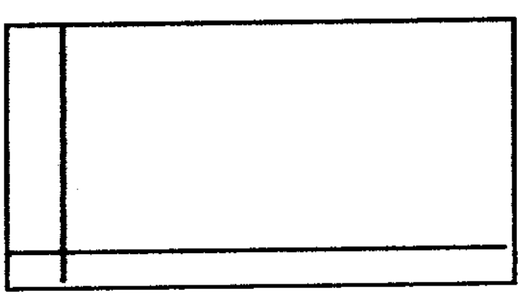
____

We are now ready to write our equation for the AREA of the triangle. Using the formula for the area of a triangle, the area **A** of the triangle (in terms of the base **B**) is:

A = ____________________________________

The independent variable is _____________ and the dependent variable is _____________.

Before we graph this function, let's see the scatterplot of the relationship between the BASE (L1) of the triangle and the AREA (L3) of the triangle. Change the set up of the scatterplot, graph it, and show your graph below. Then, graph your area function to see how it fits.



Remember, we wanted to find where to fold the paper to <u>maximize the area of the triangle</u>. Use the graph of your "area function" to answer this question. The maximum AREA of the the triangle is ______________ square inches. It occurs when the BASE of the triangle is ___________ inches, and the HEIGHT of the triangle is ____________ inches.

2. The Analytical Solution

In order to analytically find a function for the AREA **A** in terms of the BASE, again we first need to write a function for the HEIGHT **H** of the triangle in terms of the BASE **B**.

Draw a diagram in the space below of the paper with the top left corner folded down (similar to the figure on the first page). Label the base of the triangle **B**. Label the height of the triangle **H**. We need to write an expression for the length of the hypotenuse of the triangle. Your answer should be in terms of **H**, not **B**. (Hint: Unfold your paper to see how the hypotenuse relates to the height!)

The hypotenuse, in terms of the height **H**, is __________

Use the Pythagorean Theorem and write an equation which shows the relationship between these three sides.

______________ = _________________ + __________________

Using algebra, solve this equation for **H**. (It's not as hard as it looks!)

Write all decimals in the equations as fractions. **H** = ______________________________

Compare this equation with the result of your quadratic regression equation on page 2. Although they may not appear similar, if you look closely at the coefficients you will see that they are almost the same (if you did your measurements accurately).

Using this form of the HEIGHT equation, rewrite the AREA **A** function in terms of the BASE **B**. Keep your coefficients in fraction form and distribute where appropriate to simplify the form of the function.

A = ____

Sketch this function with the AREA vs. BASE scatterplot to see how this fits. (Of course it should fit well, since it is just another form of the other function.)

It's now time for our calculus! Explain how we can use the AREA function to analytically (not graphically) determine the value of the BASE that will maximize the AREA of the triangle.

Perform this calculus (and algebra) to determine the BASE that will maximize the AREA of the triangle. Compare your answer with the answer you obtained from the graphical solution.

Calculus Application Problem - Around the Corner

1. The Problem.
A steel pipe is being carried north up a north/south hallway that is 8 feet wide. At the end of the hall there is a right-angled turn into a wider hallway 12 feet wide. What is the length of the largest pipe that can be carried around the corner? Assume the pipe must remain horizontal and that it can not be tilted. We will not be concerned with the diameter of the pipe in computing a solution to the problem.

2. A Trial-and-Error Solution.
Using a piece of graph paper or engineering paper, draw a vertical hallway 8 units wide that makes a right-angled turn into a horizontal hallway 12 units wide.

First, without doing any measuring or calculations, what is your intuitive "guess" with respect to the maximum length of the pipe that will make it around the corner?

Approximately ____________ units.

With a piece of spaghetti representing the steel pipe, break small pieces off of the end until the spaghetti will be able to just fit around the corner. Try to find the length of the spaghetti to the nearest 0.5 unit. As you do this, try to determine some of the necessary properties of the pipe that will make it the "longest pipe to fit around the corner". (Hint: To help determine some of the properties, after you answer the problem, cut another piece off of the spaghetti and see how this pipe is different from the pipe that answers the problem.)

Record the length here. ____________ units. (Pipe is measured by units on the graph paper.)

What must be true about the pipe in relation to the two walls and the corner in order for us to find the length of the longest pipe that fits around the corner? ________________________________

____

____

3. The Analytical Solution.
This problem can actually be solved analytically two different ways. One procedure involves trigonometry, while the other does not. We are going to solve this problem both ways, and we'll see if our results match and correspond to the "trial-and error" solution.

a) Solution #1 (Without trigonometry)
Draw a straight line on your figure representing the pipe. It should connect the two outside walls and touch the inside corner. Now, on your figure, draw lines to extend the inside walls of the hallways to the opposite sides. Your figure should contain two **similar triangles** and a **rectangle**. Label the lengths of the segments on the outside walls that we know (8 and 12) and also the parts that we don't know (with an X for the horizontal segment and a Y for the vertical segment). If the steel pipe measures a total of **L** feet, the figure shows it divided into two parts. Label theses two parts of the pipe **L1** and **L2**.

Using the Pythagorean Theorem write an equation containing **L1** and an equation containing **L2**. Solve for **L1** and **L2** and write your expressions below.

L1 = ____________________ **L2** = ____________________

We need both of the above expressions in terms of the same variable **X**. Using the geometry of **similar triangles**, write a proportion which shows the relationship between **X** and **Y**.

Solve the equation for **Y** and complete the expression below, which shows the length of the pipe as a function of **X**.

L(X) = L1 + L2 = __________________________________

On your calculator, graph **L** and use your graph to find the **minimum value** of the length of the pipe. (Remember, although it is a minimum point on your graph, it is the maximum length of pipe that will fit around the corner!)

Sketch your graph below and label your axes to show the WINDOW that you used.

The pipe should be
__________ units.

How does your analytical answer compare to you **"trial-and-error"** answer using the piece of spaghetti?

____

Let's verify this answer using calculus. We know that local minimums (and maximums) of a function occur when ____.

On another sheet of paper, show all of the analytical work to solve the problem. Be neat! This step is a major part of this problem!

b) Solution #2 (With trigonometry)

Label the angle in the bottom left corner of the your figure θ. Another angle in your figure can also be labeled θ. Find it and label it.

Using your right triangle trigonometry relationships, write a trigonometric equation for the relationship between **L1** and θ, and an equation between **L2** and θ. (Do not use X and Y in your equations.) Solve the equations for **L1** and **L2** and write your equations below.

L1 = ____________________ **L2 =** ____________________

Again, the total length of the pipe is the sum of **L1** and **L2**. This time, however, the length **L** is a function of the angle θ. Complete the equation below.

L(θ) = L1 + L2 = ____

Again, we will solve this both **graphically** and **analytically**. On your calculator, graph L, and use your graph to find the minimum value of the length of the pipe. Of course, substitute **X** for the θ in your equation. Since we are going to apply our calculus on this result as well, leave your calculator in RADIAN mode but adjust your window to see a reasonable graph.
(What values of **X** make sense to the problem? ____________________

Sketch your graph below and label your axes to show the WINDOW that you used.

The pipe should be
__________ units.

How does this answer compare to your answer from **Solution #1**? ____________________

Again, let's verify this answer using the calculus and algebra necessary to solve the problem. Show all of your work again on the other paper.

Calculus Application Problem - Walk This Way

1. The Problem

A person walks away from a motion detector at various speeds for a period of 5 seconds. We will use the calculus concepts of Rieman sums and the Fundamental Theorem of Calculus to enhance our understanding of integral calculus and the relationship between velocity and distance functions.

2. Calculating the Distance Walked

The data we have collected and graphed is a distance vs. time scatterplot of the person as he/she walks away from the motion detector. Trace on the scatterplot to determine the starting distance and the ending distance from the motion detector. Record these values below, rounded to the nearest 0.01.

Starting distance: ________________ ft. Ending distance: ________________ ft.

Subtract these distances to find the total distance walked. Record this value below.

Total distance walked: ________________ ft. (We will come back to this!)

3. The Velocity vs. Time Data and the Velocity Function

We also have the "velocity" data in the stat/data editor of our calculator. (The vewlocity data is in L3.) Set up and graph a scatterplot of the velocity vs. time data. This scatterplot shows the velocity of the person at any time. This is the data we want to work with! The scatterplot appears to produce a pattern that can be modeled by a quadratic function, so we will attempt to find the "best fitting" function **v(t)** in the form:

$$v(t)=at^2+bt+c.$$

To do this we will use the power of the technology and perform a quadratic regression with our calculator. (See notes, if necessary, to obtain the quadratic regression equation.) Record the equation below, with the values of **a**, **b**, and **c**, rounded to 0.01.

v(t)= ____

Enter this function into **Y1** of your calculator and graph it. Hopefully it fits the data pretty well!

4. A Few Questions

Note: Turn off the scatterplot, and answer the following questions with the function **v(t)** that you found in part 3.

a. What was the initial velocity of the walker? ________________

b. On what time interval was the walker slowing down? ________________

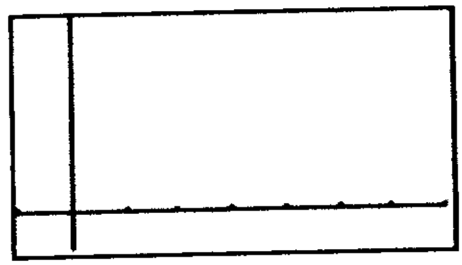
c. What was the walker's slowest speed during the 5 second period? ________________

d. What was the acceleration of the walker when t=3? ________________

5. Using the Velocity Function to Calculate the Total Distance Walked

The total distance traveled by the walker can be calculated by finding the area of the region under the velocity function and above the **t** (time) axis. We can approximate this area by dividing the region into rectangular ections and calculating the sum of the areas of the rectangles.

Sketch the graph of the velocity function below.



Our first approximation will be to divide the horizontal time axis into four equal parts. Since the total time for the data collection was 5 seconds, what is the width of each of the four rectangles?

Rectangle width = ________ secs.

Mark these values on your graph.

We will use the right endpoint of each subinterval to determine the height of each rectangle. Draw the four rectangles on your graph, and on the "top" of each rectangle write it's height (rounded to 0.01).

Finally, fill in the blanks below which show the sum of the areas of the four rectangles, and evaluate it.

Rectangle Sum = ____________ + ____________ + ____________ + ____________ = ____________

Of course, if we used more rectangles to approximate the area under the velocity function, we should get a better approximation for the total distance traveled by the walker. But, the more rectangles we use, the more work it is for us to calculate the sum of the areas of the rectangles. Unless, we can write the sum using sigma notation!

If we use 10 rectangles (of equal width) to approximate the area of the region, the width of each rectangle, which we call Δx, would be:

Rectangle width = ________ secs.

Again, if we use the right endpoint of each subinterval to determine the height of each rectangle, then the height of the first rectangle would be v(______) = ________, the height of the second rectangle would be v(_______) = ________, the third v(______) = _______, and the tenth rectangle v(________) = _________.

In general, the height of "rectangle #i" would be calculated by finding v(_________).

Now, write and evaluate an expression involving sigma, that could be used to calculate the area of the 10 rectangles.

Rectangle Sum = Σ ________________________ = ________________

Let's try one more approximation. If we use 40 rectangles, Δx = ______________ and the height of "rectangle #i" would be calculated by finding v(_________). Again, write and evaluate, an expression involving sigma, that could be used to calculate the area of the 40 rectangles.

Rectangle Sum = Σ ________________________ = ________________

Finally, our calculus tells us that if we use an "infinite number of rectangles", we should get the exact area under the curve, and, therefore, the exact distance traveled by the walker. We can not do this with our calculator but we can get a pretty god estimate by dividing the region into **100** equal subintervals, each with width $\Delta x =$ ________. The height of "rectangle #i" would be calculated by finding

v(__________).

Write and evaluate an expression involving sigma, that could be used to calculate the area of the **100** rectangles.

Rectangle Sum = Σ ____________________ = ____________________

How does this value compare to the total distance traveled by the walker in Part #2? ____________________

6. Using the Fundamental Theorem of Calculus to Calculate the Total Distance Traveled

The calculus also tells us that "limits of sums" can be calculated by writing a definite integral and evaluating it using the Fundamental Theorem of Calculus. Write and evaluate a definite integral using the Fundamental Theorem of Calculus to find the **exact area** under the velocity function v(t) from t=0 to t=5. Show your work below.

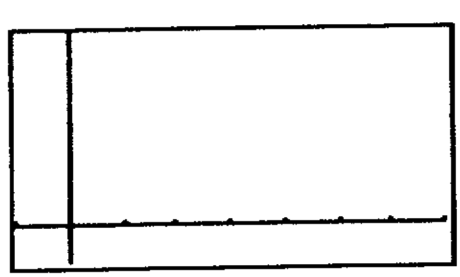
7. A Few Additional Questions

a. Normally, area values are expressed in square units, such as square feet. Notice that the areas that we have computed in this activity are in units of feet (for distance), not square feet. Explain why this is so.

b. What was the average velocity of the walker during the 5 second walk? Show how you calculate this value and include units.

c. Redraw your velocity function below, and draw a horizontal line that represents the average velocity calculated above. Applying the Mean Value Theorem to this problem says that there should be at least one time during the 5 second walk that your exact rate of change (velocity) is equal to the average rate of change (average velocity).

Using your graph, find the time(s) when the walker was traveling exactly the average velocity.



Time = ____________________

Calculus Application Problem - Don't Catch It!

1. The Problem: Imagine that you are one of many people at a party and that, unknown to everyone else, one person arrives carrying an infectious disease. How quickly will the disease spread, and what are the chances that you will leave the party with the disease?

2. Description of the Discrete Model:
The goal is to create a mathematical model that describes the spread of a disease at a party and also provides a crude description of real diseases. Suppose there are N people at the party and that the party is divided into M "stages". Let A_n represent the number of infected people at stage n, where $n=0,1,2,\ldots,M$. Let the initial condition $A_0=1$, which means that at the start of the party ($n=0$) there is one infected person. The task is to devise a rule that tells us the number of infected people at stage n+1 if we know the number of infected people at stage n. A rule that can be used to model this scenario is:

$$A_{n+1} = A_n + kA_n(N - A_n)$$, for $n=0,1,2,\ldots,M-1$, and where **k** is a proportionality constant, such that $0<k<1$.

This relationship that describes how the number of infected people evolves is called a <u>difference</u> or <u>discrete equation</u> because it involves no derivatives and determines the number of infected persons at discrete time steps.

3. An Example of the Discrete Model:
To see how this difference equation works, on another sheet of paper create a chart with headings n (Stage Number), and A_n (Number of infected persons at stage n). Let's assume that N (the number of people at the party) is 50, let k (the proportionality constant) be 0.021, and let $A_0=1$. The first row of your chart should have $n=0$ and $A_0=1$. Using the difference equation above, fill in the table to show the spread of the disease until the entire population has been infected. Note: While filling in the table, round to the nearest whole number since decimals do not make sense in this problem. On your calculator, make a scatterplot of the values in your table.

4. Description of the Continuous Model:
The previous model, which involved a difference equation, is called discrete because it gives the infected population at distinct (discrete) stages in time. The solution seems to "jump" at each stage. Another model would have the infected population increase "smoothly", or continuously. To obtain this mathematical model, we can use a <u>differential equation</u>. An argument, very similar to the one that would be used to obtain the discrete model, can be used to derive the differential equation that describes the growth of the infected population. We now let A(t) denote the number of infected people at time $t\geq0$. The corresponding differential equation is:

$$A'(t)=kA(N-A)$$, where the initial condition is $A(0)=1$, and the coefficient $k\geq0$.

Note: The constant **k** in the continuous model is not the same constant as the proportionality constant in the discrete model.

5. Solving the Differential Equation:
On another sheet of paper, by hand (without technology) solve the differential equation. (You may want to write **A'(t)** as **dA/dt.**) Show that the solution to the differential equation can be written as:

$$A(t) = \frac{N}{1+Be^{-k\cdot N\cdot t}}$$

Be sure to show all steps in your analysis.

6. Data Collection Chart: **Your ID Number:________**

Let N= the number of students in class, and let A(0)=1.

Data # 1

Stage Number	Number of Newly Infected Individuals	Number of Total Infected Individuals
0		
1		
2		
3		
4		
5		
6		

Let N=60, and let A(0)=3.

Data # 2

Stage Number	Number of Newly Infected Individuals	Number of Total Infected Individuals
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		

7. Analyzing the Data:

Use the data from the first chart.

1. Make a scatterplot of the "Stage Number" vs. the "Number of Total Infected Individuals".

2. Since the data (should) appear to be a model for a logistic function, we need to find a function in the form:

$$A(t) = \frac{c}{1 + a \cdot e^{-b \cdot t}}$$

Therefore, we need to find values for the three constants a, b, and c. The value of c should be easy. For our activity,

c= ________

To find a and b, select two other points from the table/scatterplot. (One should be the initial point, and the other should (possibly) be the next to the last point.) Substitute the ordered pairs into the equation and solve for a and b. Round your answers to two decimal places. Show your work on paper.

a= ________ b= ________ The final equation is: A(t) = ______________________________

Of course, graph it to see how it fits the scatterplot.

3. Find a logistic regression equation on your calculator, and see if it fits the data.

The regression equation is: A(t) = ______________________________

4. As another check, substitute the value of **N** (called **c** in your solution) and **k** (you need to find this from your solution above) into the differential equation **A'(t)=kA(N-A)**, and plot the **slopefield** of the differential equation on your calculator. Also plot your solution with the slopefield.

5. Using the data from the second chart, make a scatterplot of this data and repeat problem #2 above to find the logistic function that models this new data. Be sure to show your work neatly on paper.

6. How do the two models compare? How are they the same and how are they different?

Calculus Application Problem - Let It Hang!

1. The Problem: There are two functions which occur in many advanced applications of calculus and engineering. One of these applications occurs when a chain, a cable, or a power line is strung between two poles. This shape looks like a parabola, but is actually a **catenary**. We will explore these functions, look at some of their properties, their applications, and rewrite them in their Maclaurin Series representations.

2. Looking at the equation for a catenary.

A catenary can be expressed as a combination of the two functions e^x and e^{-x}. This combination can be written

as: $$y = \frac{1}{2}(e^x + e^{-x})$$

Sketch this function on your calculator with a window of [-4 , 4] by [-2 , 10] to see the basic shape of the function.

3. Finding an equation that fits a chain.

a. On a large sheet of graph paper, draw a horizontal x-axis near the bottom of the paper and a vertical y-axis down the middle of the paper.

b. Hang a chain over the paper by taping it to the upper corners of your paper. Hang it so the vertex of the chain is above the x-axis. (Is the vertex of the chain on the y-axis? It needs to be!))

c. By measuring (using the squares on the paper), find the x and y coordinates for the vertex and for the two endpoints of your chain. Try to measure accurate to the nearest 0.1 unit.

Vertex: (______ , ______) **Left endpt:** (______ , ______) **Right endpt:** (______ , ______)

d. The general equation for a hanging chain is $y = \frac{k}{2}(e^{\frac{x}{k}} + e^{-\frac{x}{k}}) + C$, where the value of k is determined by the tension in the chain and the weight of the chain.

e. Substitute the ordered pairs for the vertex and the right endpt for x and y into the general equation for the catenary to obtain two equations in terms of C and k. Simplify where obvious.

1. ______________________ 2. ______________________

f. Solve both equations for C.

C = ______________________ C = ______________________

g. Set the two equations equal to each other and use your calculator to solve for k graphically. Then find the value of C by substitution. Round answers to the nearest 0.001.

k = __________ C = __________

h. The equation of the chain is: y = ______________________

To check your results, graph the equation of your chain on your graphing calculator. What would be a good choice for your WINDOW? (Remember the piece of paper?)

Xmin __________ Xmax __________ Ymin __________ Ymax __________

(Note: If the graph does not look like the chain, you may want to select 5:ZoomSqr from the ZOOM menu.)

By TRACING, check to see if the vertex and endpts are on the function. (If not, find out what is wrong!)

i. Carefully lift the chain above the paper and tape it to the wall without removing the taped ends. (You are going to rehang the chain in the same position later).

j. Complete the table below by calculating the y-values for the given x-values of your function. Round to the nearest one decimal place.

X	±2	±4	±6	±8	±10
Y					

k. Plot these points on your paper by measuring as accurately as you can. Label your points.

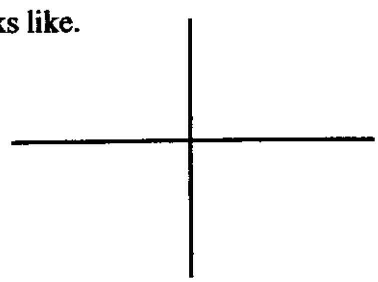
l. Rehang the chain. How closely does your calculated data fit the shape of the actual chain? (Show your instructor!)

m. Remove the chain from the wall, stretch it out straight, and measure it's length. We will use the length of the chain later in this aplication problem.

Measured value = ______________________________.

4. Defining the Hyperbolic Trigonometric Functions

Let's also consider the function formed by combining e^x and e^{-x} with the rule: $y = \frac{1}{2}(e^x + e^{-x})$. Sketch the graph of this function to see what it looks like.



Since the properties of these two functions, $y = \frac{1}{2}(e^x + e^{-x})$ and $y = \frac{1}{2}(e^x - e^{-x})$, are very similar to the properties associated with sinx and cosx, and they have a similar relationship with a hyperbola that the trig functions have with a circle, these functions are called <u>hyperbolic trigonometric functions</u> and are defined as follows:

Hyperbolic cosine of x: $\cosh x = \frac{1}{2}(e^x + e^{-x})$

Hyperbolic sine of x: $\sinh x = \frac{1}{2}(e^x - e^{-x})$

Find the "cosh" and "sinh" functions (in the CATALOG of your calculator) and sketch their graphs with their corresponding exponential forms to verify that you get the same graphs.

5. Properties of the Hyperbolic Trigonometric Functions

As was stated earlier, many of the properties of the hyperbolic trig functions are very similar to the properties of the trig functions. Below are properties of trigonometric functions (that we should be familiar with), followed by a property of a hyberbolic trig function. On engineering paper, use the <u>exponential form</u> of the hyperbolic functions to prove the stated property of the hyperbolic function. Be sure to show all of the steps in your proofs!

a. Trigonometric: $\cos^2 x + \sin^2 x = 1$

Hyperbolic: $\cosh^2 x - \sinh^2 x = 1$

b. Trigonometric: $\sin 2x = 2\sin x \cos x$

Hyperbolic: $\sinh 2x = 2\sinh x \cosh x$

c. Trigonometric: $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Hyperbolic: $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$

The hyperbolic functions also have inverse functions associated with them. It can be shown that the "inverse hyperbolic sine function" is defined as:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$ (You don't have to show this!)

d. Trigonometric: $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$

Hyperbolic: Find the derivative $\frac{d}{dx}(\sinh^{-1} x)$ and simplify it. Your final result should be very similar to the derivative of the $\sin^{-1} x$ above!

6. Maclaurin Series Representations of the Hyperbolic Functions

In this section we will find the Maclaurin Series for the two functions $y = \cosh x$ and $y = \sinh x$ by using the xponential forms and the properties that we know that apply to Maclaurin/Taylor series in general. (Do al of the following on another sheet of paper.)

a. Write the first 5 terms and the sigma notation that we have found for the series representation for e^x.

b. Use the series for e^x and write the first 5 terms and the sigma notation for the series representation for e^{-x}.

c. Use the two series above and write the first 5 terms and the sigma notation for the series representation for $e^x + e^{-x}$.

d. Write the first 5 terms and the sigma notation for the series representation for $\frac{1}{2}(e^x + e^{-x})$.

e. Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$, the answer above is the Maclaurin series for coshx. How does this compare to the Maclaurin series for the trig function cosx?

f. Using properties of Maclaurin series discussed in class, write the first 5 terms and the sigma notation for the series representation for sinhx. Explain what you are doing to produce this series.

7. Finding the Length of the Chain

The general form of the catenary $y = \frac{k}{2}(e^{\frac{x}{k}} + e^{-\frac{x}{k}}) + C$ can be written in the "cosh" form as $y = k \cdot \cosh\left(\frac{x}{k}\right) + C$.

Rewrite the specific equation of the chain found during recitation in its "cosh" form.

$y =$ ______________________________

Graph it to check that it is equivalent to the exponential form.

We derived the following formula to determine the **actual length L of a function** on an interval **[a , b]**.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

a. Find y' for the equation of the chain and simplify it.

$y' =$ ______________________________ = ______________________________

b. The expression **$1+(y')^2$** can be simplified with one of the identities you proved earlier. Rewrite the expression using this identity, and take the square root of it.

c. Write and evaluate (with your calculator) the integral expression that can be used to calculate the **length** of your chain between the **right endpt** and **left endpt**.

L = ______________________________ = ______________

How does this value compare to the length of the chain when it was measured? ______________________